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Harmonic bundles, topological-antitopological fusion and the related pluriharmonic maps ☆

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Abstract

In this work we generalize the notion of a harmonic bundle of Simpson [C.T. Simpson, Higgs-bundles and local systems, Institut des hautes Etudes Scientifiques, Publication Mathematiques, N 75 (1992) 5–95] to the case of indefinite metrics. We show, that harmonic bundles are solutions of *tt**-geometry. Further we analyze the relation between metric tt*-bundles of rank r over a complex manifold M and pluriharmonic maps from M into the pseudo-Riemannian symmetric space $\mathrm{GL}(2r, \mathbb{R})/\mathrm{O}(2p, 2q)$ in the case of a harmonic bundle. It is shown, that in this case the associated pluriharmonic maps take values in the totally geodesic subspace $GL(r, \mathbb{C})/U(p, q)$ of $GL(2r, \mathbb{R})/O(2p, 2q)$. This defines a map Φ from harmonic bundles over M to pluriharmonic maps from M to $GL(r, \mathbb{C})/U(p, q)$. Its image is also characterized in the paper. This generalizes the correspondence of harmonic maps from a compact Kähler manifold N into $GL(r, \mathbb{C})/U(r)$ and harmonic bundles over N proven in Simpson's paper [C.T. Simpson, Higgs-bundles and local systems, Institut des hautes Etudes Scientifiques, Publication Mathematiques, N 75 (1992) 5–95] and explains the link between the pluriharmonic maps related to the two geometries.

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1. Introduction

Topological–antitopological fusion or tt^*-geometry is a topic of mathematical and physical interest. In physics these geometries appeared in the context of topological quantum-field-theories [5]. Mathematically, these theories are a generalization of variations of Hodge-structures. Special geometries are particular tt^*-geometries. This follows from the variations of Hodge-structures approach of [8] and was shown directly by differential geometric arguments in [4]. The differential geometric notion of special geometry can be found in [1] and [7].

An interesting result from our point of view was the existence of a map Φ from the space of metric tt^*-bundles of rank r over the complex manifold (M, J) to the space of (twisted) pluriharmonic maps from the complex manifold (M, J) to the pseudo-Riemannian symmetric space $GL(r, \mathbb{R})/O(p, q)$, where (p, q) is the signature of the metric and the characterization of the image of Φ. In the positive definite case the map Φ is essentially bijective. For metric tt^*-bundles with positive definite metric on a real form of the holomorphic tangent bundle $T^{1,0}M$ of the manifold (M, J) this result is due to Dubrovin [6]. The generalized case was proven in [16] (compare [15] for the case of positive definite metrics).

From [15] and [8] we knew, that harmonic bundles are objects, which are closely related to tt^*-bundles. A link between these bundles and harmonic maps from compact Kähler manifolds to $GL(r, \mathbb{C})/U(r)$ was found in [17]. From Sampson's theorem [13] it follows that in this case the notion of harmonic map and pluriharmonic map coincide. Hence, this is a very similar situation to that described in the last paragraph. The question of the connection between these results arises and is discussed in this paper, which is a part of the results of the authors 'Diplomarbeit' [14] and presented here in a more general context.

We generalize the notion of a harmonic bundle by admitting indefinite metrics. With this definition we construct tt^*-bundles from harmonic bundles. To this we apply the result of [16] and prove that the target space of the pluriharmonic maps can be restricted to the totally geodesic subspace $GL(r, \mathbb{C})/U(p, q)$ of $GL(2r, \mathbb{R})/O(2p, 2q)$. The characterization of the image of Φ translates to a condition (P) and Simpson's result for positive definite signature is obtained, since for positive definite signature the map Φ is essentially bijective. Our result is a generalization (for more information see Section 4), as arbitrary signature of the bundle metric is admitted and the compactness and Kähler condition are not needed. We restrict to simply-connected manifolds M, since the case with non-tivial fundamental group can be obtained by utilizing the corresponding theorems in [16]. The pluriharmonic maps are then replaced by twisted pluriharmonic maps. This paper can been seen as the succession of [4] in a series of works, in which we study the pluriharmonic maps associated to particular solutions of tt^*-geometry.

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2. Pluriharmonic maps

Definition 1. Let (M, J) be a complex manifold and (N, h) a pseudo-Riemannian manifold. A map $f : M \to N$ is called pluriharmonic if $f|_C$ is harmonic for every complex curve $C \subset M$.

Indeed, the harmonicity of $f|_C$ is independent of the choice of a Riemannian metric in the conformal class of C, by conformal invariance of the harmonic map equation for (real) surfaces.

For a proof of the following proposition we refer to [4].

Proposition 1. *Let (M, J) be a complex manifold and (N, h) a pseudo-Riemannian manifold with Levi-Civita connection ∇^h, D a connection on M which satisfies*

$$D_{JY}X = JD_Y X \tag{2.1}$$

for all vector fields which satisfy $\mathcal{L}_X J = 0$ (i.e. for which $X - iJX$ is holomorphic), $f : (M, J) \to (N, h)$ a smooth map and ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^h.

(i) *A map $f : (M, J) \to (N, h)$ is pluriharmonic if and only if it satisfies the following equation*

$$\nabla''\partial f = 0, \tag{2.2}$$

where $\partial f = \mathrm{d}f^{1,0} \in \Gamma\left(\bigwedge^{1,0} T^*M \otimes_{\mathbb{C}} (TN)^{\mathbb{C}}\right)$ is the (1,0)-component of $\mathrm{d}f$ and ∇'' is the $(0, 1)$-component of $\nabla = \nabla' + \nabla''$.

(ii) *Any complex manifold (M, J) admits a torsion-free complex connection, i.e. a torsion-free connection D which satisfies $DJ = 0$.*

(iii) *Any torsion-free complex connection D satisfies (2.1).*

The first part of this proposition is often choosen as an alternative definition of pluriharmonic maps, see for example definition 14.2.2. in [3].

In the sequel, we need a special class of maps, which transports pluriharmonic maps into pluriharmonic maps. One knows from the theory of harmonic maps:

Proposition 2. *Let M, X, Y* be *pseudo-Riemannian manifolds and $\Psi : X \to Y$ a totally geodesic immersion. Then a map $f : M \to X$ is harmonic if and only if $\Psi \circ f : M \to Y$ is harmonic.*

and from the definition of pluriharmonic maps we obtain

Corollary 1. *Let M be a complex manifold, X, Y pseudo-Riemannian manifolds and $\Psi : X \to Y$ a totally geodesic immersion. Then a map $f : M \to X$ is pluriharmonic if and only if $\Psi \circ f : M \to Y$ is pluriharmonic.*

We want to apply this result to the symmetric spaces G/K for $G = GL(r, \mathbb{R})$ and $K = O(p, q)$ or $G = GL(r, \mathbb{C})$ and $K = U(p, q)$,[1] where $p + q = r$. We discuss this for the second example, because the first is very similar and was discussed in [16].

Let $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ be the complex hermitian $r \times r$ matrices with hermitian signature (p, q) and $I = I_{p,q} = \mathrm{diag}(\mathbb{1}_p, -\mathbb{1}_q)$.

Claim. $GL(r, \mathbb{C})$ *operates on* $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ *via*

$$GL(r, \mathbb{C}) \times \mathrm{Herm}_{p,q}(\mathbb{C}^r) \to \mathrm{Herm}_{p,q}(\mathbb{C}^r), \qquad (g, B) \mapsto g \cdot B := (g^{-1})^H B g^{-1},$$

where g^H is the hermitian conjugate of g.

The stabilizer of I is

$$GL(r, \mathbb{C})_I = \{g \in GL(r, \mathbb{C}) | g \cdot I = (g^{-1})^H I g^{-1} = I\} = U(p, q)$$

and the action is transitive due to Sylvester's theorem. This yields, by identifying orbits and rest classes, a diffeomorphism

$$\Psi : GL(r, \mathbb{C})/U(p, q) \tilde{\to} \mathrm{Herm}_{p,q}(\mathbb{C}^r) \subset GL(r, \mathbb{C}), \qquad gU(p, q) \mapsto (g^{-1})^H I g^{-1}.$$

Proposition 3. *Let (M, J) be a complex manifold. Then the map Ψ is totally geodesic and a map $\phi : M \to H(p, q) := GL(r, \mathbb{C})/U(p, q)$, where the target-space is carrying the (pseudo-)metric induced*[2] *by the Ad-invariant trace-form* (*i.e.* $A, B \mapsto \mathrm{tr}(AB)$) *on* $\mathfrak{gl}_r(\mathbb{C})$, *is pluriharmonic if and only if*

$$\psi = \Psi \circ \phi : M \to GL(r, \mathbb{C})/U(p, q) \tilde{\to} \mathrm{Herm}_{p,q}(\mathbb{C}^r) \subset GL(r, \mathbb{C})$$

is pluriharmonic.

Proof. The idea is to relate Ψ to the totally geodesic Cartan-immersion (For more information we refer to [2] theorem 3.42 and [10] volume II chapter X and XI to extend the proof of [2] to non-compact groups G. Further references are [11] and [12].). Therefore we define

$$\sigma : GL(r, \mathbb{C}) \to GL(r, \mathbb{C}), \quad g \mapsto (g^{-1})^{\dagger}.$$

[1] Here $O(p, q)$ and $U(p, q)$ are the orthogonal and unitary groups of signature (p, q).

[2] Compare [KN] volume 2, ch. X.3 and [2] proposition 3.16 for the construction of the metric on the quotient from Ad-invariant metrics on $\mathrm{herm}(p, q)$ (*see* Eq. (2.3)).

Here $g^\dagger$ denotes the adjoint of g with respect to the hermitian scalar product defined by $<\cdot,\cdot>=<I_{p,q}\cdot,\cdot>_{\mathbb{C}^r}$, where $<\cdot,\cdot>_{\mathbb{C}^r}$ is the hermitian standard scalar product on $\mathbb{C}^r$ and $I=I_{p,q}$. Explicitly it is $g^\dagger = Ig^{\mathrm{H}}I$.

σ is a homomorphism and an involution satisfying $GL(r,\mathbb{C})^\sigma = U(p,q)$.

Hence the Cartan-immersion can be written as

$$\begin{aligned} &i : GL(r,\mathbb{C})/U(p,q) \to GL(r,\mathbb{C}),\\ &g \mapsto g\sigma(g^{-1}) = gg^\dagger = gIg^H I = R_I \circ \Psi \circ \Lambda(g), \end{aligned}$$

where R_{h} is the right-multiplication with $h \in GL(r,\mathbb{C})$ and Λ the map induced on $GL(r,\mathbb{C})/U(p,q)$ by $\tilde{\Lambda} : GL(r,\mathbb{C}) \to GL(r,\mathbb{C})$, $g \mapsto (g^{-1})^{\mathrm{H}}$. Both are isometries of the invariant metric and therefore Ψ is totally geodesic. □

To be complete we mention the related symmetric decomposition:

$$\mathfrak{h} = \{h \in \mathfrak{gl}_r(\mathbb{C}) | h^\dagger = -h\} = \mathfrak{u}(p,q)$$

and

$$\mathfrak{p} = \{h \in \mathfrak{gl}_r(\mathbb{C}) | h^\dagger = h\} =: \mathrm{herm}(p,q). \tag{2.3}$$

Let $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$ be the space of symmetric $r \times r$ matrices of symmetric signature (p,q) and

$$\tilde{\Psi} : GL(r,\mathbb{R})/O(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset GL(r,\mathbb{R})$$

the identification obtained from the analogous action of $GL(r,\mathbb{R})$ on $\mathrm{Sym}_{p,q}(\mathbb{R}^r)$. With a similar argumentation we obtain (compare [16])

Proposition 4. *Let (M,J) be a complex manifold. Then the map $\tilde{\Psi}$ is totally geodesic and a map $\phi : M \to S(p,q) := GL(r,\mathbb{R})/O(p,q)$, where the target-space carries the (pseudo)metric induced by the Ad-invariant trace-form (i.e. $A, B \mapsto \mathrm{tr}(AB)$) on $\mathfrak{gl}_r(\mathbb{R})$, is pluriharmonic if and only if*

$$\psi = \tilde{\Psi} \circ \phi : M \to GL(r,\mathbb{R})/O(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset GL(r,\mathbb{R})$$

is pluriharmonic.

In this case the corresponding symmetric decomposition is:

$$\mathfrak{h} = \{h \in \mathfrak{gl}_r(\mathbb{R}) | h^{\tilde{\dagger}} = -h\} = \mathfrak{o}(p,q), \qquad \mathfrak{p} = \{h \in \mathfrak{gl}_r(\mathbb{R}) | h^{\tilde{\dagger}} = h\} =: \mathrm{sym}(p,q).$$

Here $g^{\tilde{\dagger}}$ denotes the adjoint of g with respect to the (pseudo)-scalar product defined by $<\cdot,\cdot>=<I_{p,q}\cdot,\cdot>_{\mathbb{R}^r}$, where $<\cdot,\cdot>_{\mathbb{R}^r}$ is the standard euclidian scalar product on $\mathbb{R}^r$.

3. tt*-bundles and associated pluriharmonic maps

We recall the definition of a tt^*-bundle.

Definition 2. (Compare [4] and [16]) A tt*-bundle (E, D, S) over a complex manifold (M, J) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ which satisfy the tt*-equation

$$R^\theta = 0 \quad \text{for all} \quad \theta \in \mathbb{R}, \tag{3.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + (\cos\theta)S_X + (\sin\theta)S_{JX} \quad \text{for all} \quad X \in TM. \tag{3.2}$$

A metric tt*-bundle (E, D, S, g) is a tt*-bundle (E, D, S) endowed with a possibly indefinite D-parallel fiber metric g such that for all $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all} \quad X, Y, Z \in T_{\mathrm{p}}M. \tag{3.3}$$

Remark.

(1) The flatness of the connection D^θ can be expressed in a set of equations on D and S, which can be found in [4,16].
(2) If (E, D, S) is a tt*-bundle then (E, D, S^θ) is a tt*-bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = (\cos\theta)S + (\sin\theta)S_J \tag{3.4}$$

The same remark applies to metric tt*-bundles.
(3) We want to remark further that a metric tt^*-bundle corresponds to the real-subbundle $K_{\mathbb{R}}$ of a $(D, C, \tilde{C}, \kappa, h)$ structure in [8] with the data induced on $K_{\mathbb{R}}$ by $(D, C, \tilde{C}, \kappa, h)$.

Given a metric tt*-bundle (E, D, S, g), we consider the flat connection D^θ for a fixed $\theta \in \mathbb{R}$. Any D^θ parallel frame $s = (s_1, \ldots, s_r)$ of E defines a map

$$G = G^{(s)} : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r) = \{A \in \mathrm{GL}(r) | A^{\mathrm{t}} = A \text{ has signature } (p, q)\},$$
$$x \mapsto G(x) := (g_x(s_i(x), s_j(x))), \tag{3.5}$$

where (p, q) is the signature of the metric g.

The following theorem was proven in [16]. In the case of metric tt^*-bundles with positive definite metric on a real form of the holomorphic tangent bundle $T^{1,0}M$ of the manifold (M, J) it was shown by Dubrovin [6].

Theorem 1.

1. *Let (M, J) be a simply-connected complex manifold. Let (E, D, S, g) be a metric tt^*-bundle where E has rank r and M dimension n. The representation of the metric g in a D^θ-flat frame of E $f : M \to \mathrm{Sym}_{p,q}(\mathbb{R}^r)$ induces a pluriharmonic map $\tilde{f} : M \xrightarrow{f} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \tilde{\to} S(p, q)$, where $S(p, q)$ carries the pseudo-Riemannian metric*

induced by the Ad-invariant trace-form on $\mathfrak{gl}_r(\mathbb{R})$. *Moreover, for all* $x \in M$ *the image of* $T^{1,0}M$ *under the complex linear extension of* $\mathrm{d}L_u^{-1}\mathrm{d}\tilde{f}_x : T_xM \to T_oS(p,q) = \mathrm{sym}(p,q)$ *with canonical base point o consists of commuting matrices, where* $u \in GL(r)$ *is any element such that* $\tilde{f}(x) = u \cdot o$ *and* $L_u : S(p,q) \to S(p,q)$ *is the isometry induced by the left-multiplication with* $u \in GL(r)$. *Let s' be another* D^θ*-flat frame. Then* $s' = s \cdot U$ *for a constant matrix and the pluriharmonic map associated to* s' *is* $f' = U^{\mathrm{t}} fU$.

2. *Let* (M, J) *be a simply-connected complex manifold and put* $E = M \times \mathbb{R}^r$. *Then a pluriharmonic map* $\tilde{f} : M \to S(p,q)$ *gives rise to a pluriharmonic map* $f : M \xrightarrow{\tilde{f}} S(p,q) \tilde{\to} \mathrm{Sym}_{p,q}(\mathbb{R}^r) \subset GL(r)$.

   If for all $x \in M$ *the image of* $T^{1,0}M$ *under the complex linear extension of* $\mathrm{d}L_u^{-1}\,\mathrm{d}\tilde{f}_x : T_xM \to T_oS(p,q) = \mathrm{sym}(p,q)$ *consists of commuting matrices, where* $u \in GL(r)$ *is any element such that* $\tilde{f}(x) = u \cdot o$ *and* $L_u : S(p,q) \to S(p,q)$ *is the isometry induced by the left-multiplication with* $u \in GL(r)$, *then the map f induces a metric tt*−*bundle $(E, D = \partial + S, S = -d\tilde{f}, g = < f\cdot, \cdot >_{\mathbb{R}^r})$ *on M where* ∂ *is the canonical flat connection on E.*

   In the case of signature $(r, 0)$ *and* $(0, r)$ *the map* $\tilde{f}$ *has the required property.*

Remark. It is rather surprising and non-trivial that in the case of signature $(r, 0)$ and $(0, r)$ the condition on the differential of the map $\tilde{f}$ holds. The reason for the special role of this signature can be understood by looking at the proof of [16]. There the property of the differential of the pluriharmonic map $\tilde{f}$ from M to $GL(r)/O(r)$ follows from Sampson's theorem [13] and the compactness of the group $O(r)$, more precisely from the definiteness of the metric induced by the trace-form of $GL(r)$ on $O(r)$. This argument does not work in the case of the other signatures, since the groups $O(p,q)$ in these signatures are not compact. We recall, that for metric *tt**-bundles with positive definite metric on a real form of the holomorphic tangent bundle $T^{1,0}M$ of the manifold (M, J) this result is due to Dubrovin. In the third section he uses another proof to obtain the result. The interested reader is invited to have a look at his work [6] Section 3.

4. Harmonic bundles as solutions of *tt**-geometries

In this section we introduce the notion of a harmonic bundle and show that every such bundle gives a solution of the *tt**-equations.

Definition 3. A harmonic bundle $(E \to M, D, C, \bar{C}, h)$ consists of the following data: A complex vector-bundle E over a complex manifold M, a hermitian pseudo-metric h, a metric connection D with respect to h and two C^∞- linear maps $C : \Gamma(E) \to \Gamma(T^{1,0}M \otimes E)$ and $\bar{C} : \Gamma(E) \to \Gamma(T^{0,1}M \otimes E)$, such that the connection

$$D^{(\lambda)} = D + \lambda C + \lambda^{-1}\bar{C}$$

is flat for all $\lambda \in \mathbb{S}^1$ and $h(C_X a, b) = h(a, \bar{C}_{\bar{X}} b)$ with $a, b \in \Gamma(E)$ and $X \in \Gamma(T^{1,0}M)$.

Remark. If the metric h is positive definite this definition is equivalent to the definition of a harmonic bundle in Simpson [17]. Equivalent structures with metrics of arbitrary signature have been also regarded in [8].

Theorem 2. *Let $(E \to M, D, C, \bar{C}, h)$ be a harmonic bundle over the complex manifold (M, J), then $(E, D, S, g = \mathrm{Re}\, h)$ with $S_X := C_Z + \bar{C}_{\bar{Z}}$ for $X = Z + \bar{Z} \in TM$ and $Z \in T^{1,0}M$ is a metric tt*-bundle.*

Proof. For $\lambda = \cos(\alpha) + i \sin(\alpha) \in \mathbb{S}^1$ we have a look at $D^{(\lambda)}$:

$$\begin{aligned} D^{(\lambda)}_X &= D_X + \lambda C_Z + \bar{\lambda}\bar{C}_{\bar{Z}} = D_X + \cos(\alpha)(C_Z + \bar{C}_{\bar{Z}}) + \sin(\alpha)(iC_Z - i\bar{C}_{\bar{Z}}) \\ &= D_X + \cos(\alpha)S_X + \sin(\alpha)(C_{JZ} + \bar{C}_{J\bar{Z}}) \\ &= D_X + \cos(\alpha)S_X + \sin(\alpha)S_{JX} = D^{\alpha}_X. \end{aligned}$$

Hence we have

$$D^{\alpha} = D^{(\lambda)} \tag{4.1}$$

and D^{α} is flat if and only if $D^{(\lambda)}$ is flat.

Further we claim, that S is g-symmetric. With $X = Z + \bar{Z}$ for $Z \in T^{1,0}M$ one finds

$$h(S_X \cdot, \cdot) = h(C_Z + \bar{C}_{\bar{Z}} \cdot, \cdot) = h(\cdot, C_Z + \bar{C}_{\bar{Z}} \cdot) = h(\cdot, S_X \cdot)$$

and consequently the symmetry of S with respect to $g = \mathrm{Re} h$.

Finally we show $Dg = 0$:

$$\begin{aligned} 2X.g(e, f) &= X.(h(e, f) + h(f, e)) = (Z + \bar{Z}).(h(e, f) + h(f, e)) \\ &= h(D_Z e, f) + h(e, D_{\bar{Z}} f) + h(D_{\bar{Z}} e, f) + h(e, D_Z f) \\ &\quad + h(D_Z f, e) + h(f, D_{\bar{Z}} e) + h(D_{\bar{Z}} f, e) + h(f, D_Z e) \\ &= h((D_Z + D_{\bar{Z}})e, f) + h(e, (D_{\bar{Z}} + D_Z) f) + h((D_Z + D_{\bar{Z}}) f, e) \\ &\quad + h(f, (D_{\bar{Z}} + D_Z)e) \\ &= h(D_X e, f) + h(e, D_X f) + h(D_X f, e) + h(f, D_X e) \\ &= 2(g(D_X e, f) + g(e, D_X f)). \end{aligned}$$

This proves, that $(E, D, S, g = \mathrm{Re}\, h)$ is a metric tt^*-bundle. □

Remark. Here we have taken the underlying real bundle of a harmonic bundle to obtain a tt^*-bundle. In this sense one can see a harmonic bundle as a special case of a tt^*-bundle. On the other hand, one can interpret a $(D, C, \tilde{C}, \kappa, h)$ structure as a harmonic bundle $(D, C, \tilde{C}, h)$ by forgetting κ. This means that both can be understood as special cases of each other.

5. The pluriharmonic map associated to a harmonic bundle

In the last section we have shown, that every harmonic bundle induces a metric tt^*-bundle and hence a pluriharmonic map to $S(2p, 2q) = Gl(2r, \mathbb{R})/O(2p, 2q)$ where (p, q) is the hermitian signature of h. Later in this section, we use the additional information of the harmonic bundle structure to restrict the target of the pluriharmonic map to $H(p, q) = GL(r, \mathbb{C})/U(p, q)$. At the end of the chapter we get a correspondence. Collecting our current knowledge we obtain the corollary:

Corollary 2. *Let $(E \to M, D, C, \bar{C}, h)$ be a harmonic bundle over the simply-connected complex manifold (M, J), then the representation of $g = \mathrm{Re}\, h$ in a $D^{(\lambda)}$-flat frame defines a pluriharmonic map $\Phi_g : M \to S(2p, 2q)$.*

Proof. This follows from the identity (4.1), i.e. $D_X^{(\lambda)} = D_X^\alpha$ for $\lambda = \cos(\alpha) + i \sin(\alpha) \in \mathbb{S}^1$ and from Theorem 1. □

To restrict the image of Φ_g we have a look at taking the real-part of h.

In the following text we identify $\mathbb{C}^r$ with $\mathbb{R}^r \oplus i\mathbb{R}^r = \mathbb{R}^{2r}$. In this model the multiplication with i coincides with the automorphism

$$j = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$$

and $GL(r, \mathbb{C})$ (respectively $\mathfrak{gl}_r(\mathbb{C})$) are the elements in $GL(2r, \mathbb{R})$ (respectively $\mathfrak{gl}_{2r}(\mathbb{R})$), which commute with j. An endomorphism $C \in \mathrm{End}(\mathbb{C}^r)$ decomposes in its real-part A and its imaginary part B, i.e. $C = A + iB$ with $A, B \in \mathrm{End}(\mathbb{R}^r)$. In the above model C is given by the matrix

$$C \leftrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

The complex conjugated of C is

$$\bar{C} \leftrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

the transpose $C^{\mathrm{t}} = A^{\mathrm{t}} + iB^{\mathrm{t}}$

$$C^{\mathrm{t}} \leftrightarrow \begin{pmatrix} A^{\mathrm{t}} & -B^{\mathrm{t}} \\ B^{\mathrm{t}} & A^{\mathrm{t}} \end{pmatrix}$$

and consequently the hermitian conjugated is

$$\bar{C}^{\mathrm{t}} \leftrightarrow \begin{pmatrix} A^{\mathrm{t}} & B^{\mathrm{t}} \\ -B^{\mathrm{t}} & A^{\mathrm{t}} \end{pmatrix}.$$

We observe, that $\bar{C}^t = C^{\mathrm{T}}$ where $\cdot^{\mathrm{T}}$ is the transpose in $\mathrm{End}(\mathbb{R}^{2r})$.

The hermitian matrices $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ of signature (p, q) coincide with subset of symmetric matrices $H \in \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$, which commute with j, i.e. $[H, j] = 0$. Likewise, $T_{I_{p,q}}\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ consists of symmetric matrices $h \in \mathrm{sym}(\mathbb{R}^{2r})$, which commute with j, i.e. the hermitian matrices in $\mathfrak{gl}_{2r}(\mathbb{R})$ which we denote by $\mathrm{herm}_{p,q}(\mathbb{C}^r)$.

A hermitian scalar-product h of signature (p, q) corresponds to a hermitian matrix $H \in \underline{Herm}_{p,q}(\mathbb{C}^r)$ of hermitian signature (p, q) defined by $h(\cdot, \cdot) = (H\cdot, \cdot)_{\mathbb{C}^r}$. The condition $\bar{C}^{\mathrm{t}} = C$, i.e. C hermitian, means in our model, that C has the form

$$C \leftrightarrow \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with $A = A^{\mathrm{t}}$ and $B = -B^t$.

Finally we find the explicit representation of the map $\mathcal{R}$, which corresponds to taking the real-part of the hermitian metric, i.e. $\mathrm{Re}h = (\mathcal{R}(H)\cdot, \cdot)_{\mathbb{R}^{2r}}$:

$$\mathcal{R} : \mathrm{Herm}_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r}), \qquad H \mapsto \tfrac{1}{2}(H + \bar{H}^{\mathrm{t}}) = \tfrac{1}{2}(H + H^{\mathrm{T}}) = \iota(H),$$

where ι is the canonical inclusion $\mathrm{Herm}_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$. This map has maximal rank and is equivariant with respect to $GL(r, \mathbb{C})$.

Further we claim, that it is totally geodesic: The decomposition $\mathfrak{gl}_{2r}(\mathbb{R}) = \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r}) \oplus \mathfrak{o}(2p, 2q)$ is a symmetric decomposition of the symmetric space $GL(2r)/O(2p, 2q)$ and hence[3]

$$[[\mathrm{sym}_{2p,2q}(\mathbb{R}^{2r}), \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r})], \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r})] \subset \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r}).$$

From $[A, j] = [B, j] = [C, j] = 0$, we conclude with the Jacobi identity $[[A, B], j] = 0$ and $[[[A, B], C], j] = 0$. Consequently $T_{I_{p,q}}\mathrm{Herm}_{p,q}(\mathbb{C}^r) = \mathrm{herm}_{p,q}(\mathbb{C}^r)$ is a Lie-triple-product[3] in $T_{I_{p,q}}\mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r}) = \mathrm{sym}_{2p,2q}(\mathbb{R}^{2r})$, i.e.

$$[[\mathrm{herm}_{p,q}(\mathbb{C}^r), \mathrm{herm}_{p,q}(\mathbb{C}^r)], \mathrm{herm}_{p,q}(\mathbb{C}^r)] \subset \mathrm{herm}_{p,q}(\mathbb{C}^r)$$

and finally $\mathcal{R} : \mathrm{Herm}_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$ is a totally geodesic map.

[3] We refer to [9] Ch. IV.7, [10] vol. 2, ch. XI.4 and [11] ch. III for more informations on Lie-triple-products and totally geodesic subspaces of symmetric spaces and [10] vol. 2, ch. XI.2 for the (canonical) symmetric decomposition of a symmetric space.

Summarizing we have the commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{h} & \mathrm{Herm}_{p,q}(\mathbb{C}^r) \subset GL(r,\mathbb{C}) & \xrightarrow{\mathcal{R}} & \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r}) \subset GL(2r,\mathbb{R}) \\ {\scriptstyle id_M}\downarrow & & {\scriptstyle \Psi^{-1}}\downarrow & & {\scriptstyle \tilde{\Psi}^{-1}}\downarrow \\ M & \xrightarrow{\tilde{h}} & GL(r,\mathbb{C})/U(p,q) & \xrightarrow{[i]} & GL(2r,\mathbb{R})/O(2p,2q), \end{array}$$

where $[i]$ is induced by the inclusion $i : GL(r, \mathbb{C}) \hookrightarrow GL(2r, \mathbb{R})$. All maps in the right square of this diagram are totally geodesic. This gives the proposition:

Proposition 5. *A map $h : M \to \mathrm{Herm}_{p,q}(\mathbb{C}^r)$ is pluriharmonic, if and only if $g = \mathrm{Re}h : M \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$ is pluriharmonic. A map $\tilde{h} : M \to H(p, q)$ is pluriharmonic, if and only if $\tilde{g} = [i] \circ h : M \to S(2p, 2q)$ is pluriharmonic.*

Proof. As discussed above the map $\mathcal{R} : \mathrm{Herm}_{p,q}(\mathbb{C}^r) \to \mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$ is totally geodesic and an immersion. This means that we are in the situation of Corollary 1.

The second claim follows from the commutative diagram on the right hand-side and the statements of Proposition 3 and Proposition 4, that the composition of a map f from M to $\mathrm{Herm}_{p,q}(\mathbb{C}^r)$ (respectively $\mathrm{Sym}_{2p,2q}(\mathbb{R}^{2r})$) with Ψ^{-1} (respectively $\tilde{\Psi}^{-1}$) is pluriharmonic, if and only if f is pluriharmonic. □

Theorem 3. *Let $(E \to M, D, C, \bar{C}, h)$ be a harmonic bundle over the simply-connected complex manifold (M, J). Then the representation of h in a $D^{(\lambda)}$-flat frame defines a pluriharmonic map $\phi_h : M \to \mathrm{Herm}_{p,q}(\mathbb{C}^r)$. In addition, for all $x \in M$ the image of $T^{1,0}M$ under the complex linear extension of* $\mathrm{d}L_u^{-1} d(\tilde{\phi}_h)_x : T_xM \to T_oH(p, q) = \mathrm{herm}(p, q)$ consists of commuting matrices, where u is an arbitrary element satisfying $\tilde{\phi}_h(x) = u \cdot o$, o is the canonical base point and $L_u : H(p, q) \to H(p, q)$ is the isometry induced by the left-multiplication on $GL(r, \mathbb{C})$.

Proof. The pluriharmonicity of ϕ_h follows from Corollary 2 and Proposition 5. For the second part we observe, that the differential of $\mathcal{R} : \mathfrak{gl}_r(\mathbb{C}) \to \mathfrak{gl}_{2r}(\mathbb{R})$ is a homomorphism of Lie-algebras and therefore preserves the vanishing of the Lie-bracket. □

The following theorem gives the converse statement

Theorem 4. *Let (M, J) be a simply-connected complex manifold and $E = M \times \mathbb{C}^r$. A pluriharmonic map $\tilde{\phi}_h : M \to H(p, q)$ induces a pluriharmonic map $\tilde{\phi}_g = [i] \circ \tilde{\phi}_h : M \to S(2p, 2q)$. Suppose, that for all $x \in M$ the image of $T^{1,0}M$ under the complex linear extension of* $\mathrm{d}L_u^{-1} d(\tilde{\phi}_h)_x : T_xM \to T_oH(p, q) = \mathrm{herm}(p, q)$ consists of commuting matrices, where u is an arbitrary element satisfying $\tilde{\phi}_h(x) = u \cdot o$ and $L_u : H(p, q) \to H(p, q)$ is the isometry induced by the left-multiplication on $GL(r, \mathbb{C})$. Then $(E, D = \partial + C + \bar{C}, C = -(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r})$ defines a harmonic bundle, where ∂ is the complex linear extension on TM^c of the flat connection on $E = M \times \mathbb{C}^r$. In the case of signature $(r, 0)$ and $(0, r)$ the map $\tilde{\phi}_h$ has the required property.

Proof. Due to Proposition 5 $\tilde{\phi}_g$ is pluriharmonic. Hence one obtains from Theorem 1 a tt^*-bundle $(E = M \times \mathbb{R}^{2r}, D = \partial + S, S = -d\tilde{\phi}_g, g =< \phi_g \cdot, \cdot >_{\mathbb{R}^{2r}})$, since the condition on

$d(\tilde{\phi}_g)_x$ is obtained as in Theorem 3. We are now going to use the additional information, we have from the fact, that the map ϕ_g comes from ϕ_h, to show that $(E, D = \partial + C + \overline{C}, C = -(d\tilde{\phi}_h)^{1,0}, h = (\phi_h \cdot, \cdot)_{\mathbb{C}^r})$ is a harmonic bundle.

The hermitian metric h is given by

$$h = g + \sqrt{-1}\omega$$

with $\omega = g(j\cdot, \cdot)$. This is the standard relation between hermitian metrics on complex vector spaces and the hermitian metrics on the underlying real vector spaces.

We observe $Dj = [\partial + S, j] = [S_X, j] = 0$, because S is is the derivation of a map from M to $GL(r, \mathbb{C})$ and hence commutes with j. Therefore $D\omega = 0$ follows from $Dg = 0$ and $Dh = 0$ from $D\omega = 0$ and $Dg = 0$.

From the definition of S and S_J in Theorem 2

$$S_X = C_Z + \overline{C}_{\bar{Z}},$$

$$S_{JX} = C_{JZ} + \overline{C}_{J\bar{Z}}$$

for $X = Z + \overline{Z}$ and $Z \in T^{1,0}M$ we obtain the definition of

$$2C_Z = S_X - jS_{JX},$$

$$2\overline{C}_{\bar{Z}} = S_X + jS_{JX}.$$

In addition we have the identity $D_X^{(\lambda)} = D_X^{\alpha}$ for $\lambda = \cos(\alpha) + i\sin(\alpha) \in \mathbb{S}^1$ which again gives the equivalence between the flatness of $D^{(\lambda)}$ and D^{α}.

It remains to show

$$h(C_Z\cdot, \cdot) = h(\cdot, \overline{C}_{\bar{Z}}\cdot).$$

We recall the relations $j^*g = g$ and $(*)\, g(j\cdot, \cdot) = -g(\cdot, j\cdot)$, which implies the anti-symmetry of $\omega = g(j\cdot, \cdot)$ and $(*')\, \omega(j\cdot, \cdot) = -\omega(\cdot, j\cdot)$. Further we use the identities $(**)\, [S, j] = [S_J, j] = 0$ and that $(***)\, S, S_J$ are g-symmetric. Due to (**) and (***) $(****)\, S, S_J$ ω-symmetric. These identities imply

$$\begin{aligned} 2h(C_Z\cdot, \cdot) &= g(S_X - jS_{JX}\cdot, \cdot) + i\omega(S_X - jS_{JX}\cdot, \cdot) \\ &\overset{(*),(**),(***)}{=} g(\cdot, S_X + jS_{JX}\cdot) + i\omega(S_X - jS_{JX}\cdot, \cdot) \\ &\overset{(*'),(**),(****)}{=} g(\cdot, S_X + jS_{JX}\cdot) + i\omega(\cdot, S_X + jS_{JX}\cdot) = 2h(\cdot, \overline{C}_{\bar{Z}}\cdot). \end{aligned}$$

Using $S = -d\tilde{\phi}_g = -d([i] \circ \tilde{\phi}_h) = -d\tilde{\phi}_h$ we find extending S on TM^c to S^c for $Z \in T^{1,0}M$ the equations $C_Z = S_Z^c = -d\tilde{\phi}_h(Z)$ and $\overline{C}_{\bar{Z}} = -d\tilde{\phi}_h(\overline{Z})$. □

In [17] Section 1 Simpson studied Higgs-bundles with harmonic positive definite metrics, i.e. harmonic bundles, over a compact Kähler-manifold M^n and related these to harmonic maps from M in $GL(n,\mathbb{C})/U(n)$. From his results one can find, that a given flat bundle with a harmonic metric induces a harmonic map from M in $GL(n,\mathbb{C})/U(n)$. Conversely, a harmonic map from M in $GL(n,\mathbb{C})/U(n)$ and a flat bundle give rise to a harmonic bundle. From Sampson's theorem [13] one obtains, that in the above case the notion of harmonic and pluriharmonic coincide.

This result follows from Theorems 3 and 4, since the condition on the differential of $\tilde{\phi}_h$ is satisfied in the case of signature $(r, 0)$ and $(0, r)$. We remark, that Theorems 3 and 4 are in fact more general, since the compactness of M and Kähler condition are not needed. Simpson uses Kähler-identities for vector bundles over compact Kähler manifolds in his proof. Therefore one cannot use his proof neither in the non-compact nor non-Kähler-case. Further he needs compactness, since he uses arguments from harmonic map theory, which are developped from Sius Bochner formula for harmonic maps to obtain the vanishing of the object which he calls pseudocurvature and which is the integrability constraint for a flat bundle to define a Higgs bundle. The works [6,16] deal with pluriharmonic maps and prove the results by direct calculations using the pluriharmonic and the tt^*-equations, respectively. In the case of signature $(r, 0)$ and $(0, r)$ [16] needs only the second statement of Sampson's theorem [13] (compare the remark after Theorem 1) and so compactness is not needed.

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